

BL-RINGS

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ABSTRACT. The main goal of this article is to introduce BL-rings, i.e., commutative rings whose lattices of ideals can be equipped with a structure of BL-algebra. We obtain a description of such rings, and study the connections between the new class and well known classes such as multiplication rings, Baer rings, Dedekind rings.

Key words: Multiplication ring, Baer ring, subdirectly irreducible ring, BL-ring.

1. INTRODUCTION

Given any ring (commutative or not, with or without unity) R generated by idempotents, the semiring of ideals of R under the usual operations form a residuated lattice $A(R)$. In recent articles, several authors have investigated classes of rings for which the residuated lattice $A(R)$ an algebra of a well-known subvariety of residuated lattices. For instance, rings R for which $A(R)$ is an MV-algebra, also called Łukasiewicz rings are investigated in [2], rings R for which $A(R)$ is a Gödel algebra, also called Gödel rings are investigated in [3], and very recently rings R for which $A(R)$ is an pseudo MV-algebra, also called Generalized Łukasiewicz rings are investigated in [11].

In the same spirit, the goal of the present article is we introduce and investigate the class of commutative rings R for which $A(R)$ is a BL-algebra, also referred to as BL-rings. Among other things, we show that this class is properly contained in the class of multiplication rings as treated in [9], and contains properly each of the classes of Dedekind domains, Łukasiewicz rings, discrete valuation rings, Noetherian multiplication rings. We also prove that BL-rings are closed under finite direct products, arbitrary direct sums, and homomorphic images. Furthermore, a description of subdirectly irreducible BL-rings is obtained, which combined with the well known Birkhoff theorem, provides a representation of general BL-rings.

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We recall that a commutative integral residuated lattice can be defined as a nonempty set L with four binary operations $\wedge, \vee, \otimes, \rightarrow$, and two constants $0, 1$ satisfying: (i) $\mathbb{L}(L) := (L, \wedge, \vee, 0, 1)$ is a bounded lattice; (ii) $(L, \otimes, 1)$ is a commutative monoid; and (iii) (\otimes, \rightarrow) form an adjunct pair, i.e., $x \otimes y \leq z$ iff $x \leq y \rightarrow z$, for all $x, y, z \in L$.

An *Rl monoid* is a residuated lattice L satisfying the divisibility axiom, that is: $x \wedge y = (x \rightarrow y) \otimes x$, for all $x, y \in L$.

A *BL-algebra* is an *Rl monoid* L satisfying the pre-linearity axiom, that is: $(x \rightarrow y) \vee (y \rightarrow x) = 1$, for all $x, y \in L$.

An *MV-algebra* is a BL-algebra L satisfying the double-negation law: $x^{**} = x$, for all $x \in L$, where $x^* = x \rightarrow 0$.

Given any commutative ring R generated by idempotents (that is for every $x \in R$, there exists an idempotent $e \in R$ such that $ex = x$), the lattice of ideals of R form a residuated lattice $A(R) := \langle \text{Id}(R), \wedge, \vee, \otimes, \rightarrow, \{0\}, R \rangle$, where $I \wedge J = I \cap J$, $I \vee J = I + J$, $I \otimes J := I \cdot J$, $I \rightarrow J := \{x \in R : xI \subseteq J\}$. Note that I^* is simply the annihilator of I in R .

The following notations will be used throughout the paper. Given a commutative ring R , recall that an ideal I of R is called an annihilator ideal (resp. a dense ideal) if $I = J^*$ for some ideal J of R (resp. $I^* = 0$).

$A(R)$ denotes the residuated lattice of ideals of R ;

$MV(R)$ denotes the set of annihilator ideals of R ;

$D(R)$ denotes the set of dense ideals of R .

2. BL-RINGS, DEFINITIONS, EXAMPLES AND FIRST PROPERTIES

In this section, we introduce the notion of BL-rings. As announced, these should be rings R for which $A(R)$ is naturally equipped with a BL-algebra structure. Some of the main properties of these rings, and their connections to other known classes of rings are established.

Definition 2.1. A commutative ring R is called a BL-ring if for all ideals I, J of R ,

BLR-1: $I \cap J = I \cdot (I \rightarrow J)$,

BLR-2: $(I \rightarrow J) + (J \rightarrow I) = R$.

Note that BLR-1 is equivalent to $I \cap J \subseteq I \cdot (I \rightarrow J)$ since the inclusion $I \cdot (I \rightarrow J) \subseteq I \cap J$ holds in any ring. In addition, BLR-2 is easily seen to be equivalent to each of the following conditions:

BLR-2.1: $(I \cap J) \rightarrow K = (I \rightarrow K) + (J \rightarrow K)$ for all ideals I, J, K of R .

BLR-2.2: $I \rightarrow (J + K) = (I \rightarrow J) + (I \rightarrow K)$ for all ideals I, J, K of R .

Recall [9] that a commutative ring is called a multiplication ring if every ideals I, J of R such that $I \subseteq J$, there exists an ideal K of R such that $I = J \cdot K$.

Proposition 2.2. *A commutative ring satisfies BLR-1 if and only if it is a multiplication ring.*

Proof. Suppose that R is a BL-ring and let I, J be ideals of R such that $I \subseteq J$. Then by BLR-1, $I = I \cap J = I \cdot (I \rightarrow J)$. Take $K = I \rightarrow J$.

Conversely, suppose that R is a multiplication ring and let I, J be ideals of R . Since $I \cap J \subseteq I$, there exists an ideal K of R such that $I \cap J = I \cdot K$. Hence, $I \cdot K \subseteq J$ and it follows that $K \subseteq I \rightarrow J$. Thus, $I \cap J \subseteq I \cdot (I \rightarrow J)$. As observed above, the inclusion $I \cdot (I \rightarrow J) \subseteq I \cap J$ holds in any ring. Therefore, BLR-1 holds as needed. \square

Recall that a ring R is said to be generated by idempotents if for every $x \in R$, there exist $e = e^2 \in R$ such that $xe = x$.

Corollary 2.3. 1. *Every BL-ring is generated by idempotents.*
2. *A commutative ring is a BL-ring if and only if $A(R)$ is a BL-algebra.*

Proof. 1. BLR-1 implies that R is a multiplication ring, and it is known that every multiplication ring satisfies the condition [9, Cor. 7].

2. This is clear from (1) and the axioms BLR-1 and BLR-2. \square

Example 2.4. 1. Discrete valuation rings (dvr).

The ideals of a dvr are principal and totally ordered by the inclusion. Clearly BLR-2 holds in any chain ring. As for BLR-1, let I, J be ideals of a dvr R . If $I \subseteq J$, then $I \rightarrow J = R$ and $I \cap J = I \cdot (I \rightarrow J)$. On the other hand, if $J \subseteq I$, since I is principal, then $I = aR$ for some $a \in R$. Let $j \in J \subseteq aR$, then $j = ax$. So $ax \in J$ and $x \in I \rightarrow J$. Hence, $j \in I \cdot (I \rightarrow J)$ and $I \cap J \subseteq I \cdot (I \rightarrow J)$.

2. Noetherian multiplication rings.

By Proposition 2.2, every multiplication ring satisfies BLR-1. In addition, if R is a Noetherian multiplication ring, then R is a Noetherian arithmetical ring and by [10, Thm. 3], $K \rightarrow (I + J) = (K \rightarrow I) + (K \rightarrow J)$ for all ideals I, J, K of R . Hence, R satisfies BLR-2.2, which is equivalent to BLR-2 as observed earlier. Whence, R is a BL-ring as claimed.

3. Łukasiewicz rings. Indeed, if R is a Łukasiewicz ring, then its ideals form an MV-algebra [2].

4. Gödel rings. Indeed, if R is a Gödel ring, its ideals form a BL-algebra in which $I \cdot J = I \cap J$ [3].

Remark 2.5. 1. Each of the classes of rings of Example 2.4 is a proper subclass of BL-rings. In fact \mathbb{Z} is a Noetherian multiplication ring

that is neither a dvr nor a Łukasiewicz ring. In addition, $\bigoplus_{n=1}^{\infty} \mathbb{R}$ is a Łukasiewicz ring that is neither Noetherian, nor a dvr.

2. A Noetherian ring is a BL-ring if and only if it is a multiplication ring. In addition, note that Noetherian multiplication rings are ZPI-rings (see for e.g., [12, Exercise 10(b) pp. 224]), and ZPI-rings are direct sum of a finite number of Dedekind domains and special primary rings. Therefore, Noetherian BL-rings are direct sum of a finite number of Dedekind domains and special primary rings.

Recall that a Baer ring is a ring in which every annihilator ideal is generated by an idempotent, i.e., for every ideal I of R , there exists an idempotent $e \in R$ such that $I^* = eR$.

The following lemma will be needed when working with BL-rings.

Lemma 2.6. *Let R be a ring, and I, J, K be ideals of R such that $I \subseteq J, K$. Then,*

- (a) $I \subseteq (I^* \cdot J)^*, J \rightarrow I, J \rightarrow K, K \rightarrow J$;
- (b) $(J/I)^* = (J \rightarrow I)/I$;
- (c) $(J/I) \rightarrow (K/I) = (J \rightarrow K)/I$.

Proof. These are easily derived from the definitions of the operations involved. \square

Note that if a ring satisfies BLR-2, then since $I \rightarrow J = I \rightarrow (I \cap J)$, it must satisfy the following.

BLR-3: $I \cap J = 0$ implies $I^* + J^* = R$.

Proposition 2.7. *A ring R satisfies BLR-2 if and only if every quotient (by an ideal) of R satisfies BLR-3.*

Proof. Suppose that R satisfies BLR-2, and let I be an ideal of R . Let $I \subseteq J, K$ such that $(J/I) \cap (K/I) = I$. Then, $J \cap K = I$. Now, $(J/I)^* + (K/I)^* = (J \rightarrow I)/I + (K \rightarrow I)/I = (J \rightarrow (J \cap K))/I + (K \rightarrow (J \cap K))/I = ((J \rightarrow K) + (K \rightarrow J))/I = R/I$. Thus, R/I satisfies BLR-3. Conversely, suppose that every factor of R satisfies BLR-3. Let I, J be ideals of R , then $R/(I \cap J)$ satisfies BLR-3. Since $(I/(I \cap J)) \cap (J/(I \cap J)) = I \cap J$, then $(I/(I \cap J))^* + (J/(I \cap J))^* = R/(I \cap J)$. That is, $(I \rightarrow (I \cap J))/I + (J \rightarrow (I \cap J))/I = R/(I \cap J)$, or $(I \rightarrow J)/(I \cap J) + (J \rightarrow I)/(I \cap J) = R/(I \cap J)$. Thus, $((I \rightarrow J) + (J \rightarrow I))/(I \cap J) = R/(I \cap J)$ and it follows that $(I \rightarrow J) + (J \rightarrow I) = R$. So, R satisfies BLR-2 as needed. \square

Corollary 2.8. *Every multiplication ring satisfies BLR-2 if and only if every multiplication ring satisfies BLR-3.*

The following example, which is a special case of [9, Ex. 4] is a multiplication ring that does not satisfy BLR-2.

Example 2.9. Let F be any field and let R be the subring of $\prod_{k=1}^{\infty} F$ generated by $\oplus_{k=1}^{\infty} F$ and the constant functions from $\mathbb{N} \rightarrow F$.

Then R is a multiplication ring with identity.

Note that R is the subring of $\prod_{k=1}^{\infty} F$ of all sequences $\mathbb{N} \rightarrow F$ that are eventually constant. That is $f \in R$ if and only if there exists $x \in F$ and $n \geq 1$ such that $f(k) = x$ for all $k \geq n$.

Now, let $I = \{f \in R : f(2k) = 0 \text{ for all } k \in \mathbb{N}\}$ and $J = \{f \in R : f(2k+1) = 0 \text{ for all } k \in \mathbb{N}\}$.

Then $I, J \subseteq \oplus_{k=1}^{\infty} F$, $I \cap J = 0$, $I^* = J$, $J^* = I$. Thus, $I^* + J^* = I + J \neq R$.

Proposition 2.10. *Let R be a ring that is generated by idempotents and P be a prime ideal of R . Recall [12, §IX.4] that*

$$N(P) = \{x \in R : xs = 0 \text{ for some } s \in R \setminus P\}$$

Then $\bigcap_P N(P) = 0$.

Proof. Let $x \neq 0$, then $(xR)^* \neq R$. Thus, there exists a prime ideal P of R such that $(xR)^* \subseteq P$. We claim that $x \notin N(P)$. Indeed, if $x \in N(P)$, then $xs = 0$ for some $s \notin P$. This would imply that $s \in (xR)^* \subseteq P$, which is a contradiction. \square

Proposition 2.11. *Every multiplication ring with unity is a subring of a direct product of dvrs and SPIRs.*

Proof. Let R be multiplication ring with unity. Consider $\varphi : R \rightarrow \prod_P R_P$ defined by $\varphi(x) = (\frac{x}{1})_P$. Then φ is a ring homomorphism and $\ker \varphi = \bigcap_P N(P)$. Thus, φ is injective by Proposition 2.10. On the other hand, R is an AM-ring [5, Lemma 2.4]. Therefore, by [12, Thm. 9.23, Prop. 9.25, Prop. 9.26], each R_P is either a dvr or an SPIR. \square

Proposition 2.12. *BL-rings are closed under each of the following operations.*

1. *Finite direct products;*
2. *Arbitrary direct sums;*
3. *Homomorphic images.*

Proof. 1. Let $R = \prod_{k=1}^n R_k$, where each R_k is a BL-ring. Using the fact that each R_k is generated by idempotents, one gets that any ideal I of R is of the form $I = \prod_{k=1}^n I_k$, where I_k is an ideal of R_k for all k .

On the other hand, if $I = \prod_{k=1}^n I_k$ and $J = \prod_{k=1}^n J_k$, one can easily verify the

following identities: $I \cdot J = \prod_{k=1}^n I_k \cdot J_k$, $I \rightarrow J = \prod_{k=1}^n I_k \rightarrow J_k$, $I \cap J = \prod_{k=1}^n I_k \cap J_k$, and $I + J = \prod_{k=1}^n I_k + J_k$.

From these identities, it becomes clear that R satisfies BLR-1 and BLR-2 since each R_k does. Therefore, R is a BL-ring.

2. This is very similar to the above. Indeed, one proves that ideals of direct sums of BL-rings are direct sums of ideals and the argument goes through as in (1), with each instance of the finite direct product replaced by a direct sum.

3. Let R be a BL-ring and I be an ideal of R . We shall show that R/I is a BL-ring. Recall that ideals of R/I are of the form J/I , where J is an ideal of R with $I \subseteq J$. Let J, K be ideals of R containing I . Using the properties stated in Proposition 2.6, we have $(J/I) \cap (K/I) = (J \cap K)/I = (J \cdot (J \rightarrow K))/I = (J/I) \cdot (J \rightarrow K)/I = (J/I) \cdot ((J/I) \rightarrow (K/I))$. Thus, R/I satisfies BLR-1. The verification of BLR-2 is similar. Therefore, R/I is a BL-ring as needed. \square

The following examples shows that an arbitrary direct product of BL-rings needs not a BL-ring.

Example 2.13.

3. CONNECTION WITH BAER RINGS AND VON NEUMANN RING

Proposition 3.1. *A reduced ring with identity satisfies BLR-3 if and only if it is a Baer ring.*

Proof. Suppose R is a reduced ring with identity that satisfies BLR-3. Let I be an ideal of R , then since R is reduced, $I \cap I^* = 0$. It follows from BLR-3 that $I^* + I^{**} = R$. Hence, $1 = a + b$ for some $a \in I^*$ and $b \in I^{**}$. Thus, $a = a.1 = a(a + b) = a^2 + ab = a^2$ and a is idempotent. Now, for every $x \in I^*$, $x = x.1 = x(a + b) = xa + xb = xa$. Therefore, $I^* = aR$ and R is a Baer ring.

Conversely, suppose that R is a Baer ring and let I, J be ideals such that $I \cap J = 0$. Then $I \subseteq J^* = eR$, for some idempotent $e \in R$. Note that since $I \subseteq eR$ and e is idempotent, then $1 - e \in I^*$. Thus, $1 = (1 - e) + e \in I^* + I^{**}$ and $I^* + I^{**} = R$. \square

Recall that in every Baer ring R , a^* is the unique idempotent element in R such that $(aR)^* = a^*R$. An ideal I of a Baer ring is called a Baer-ideal if for every $a, b \in R$ such that $a - b \in I$, then $a^* - b^* \in I$.

Corollary 3.2. *For every Baer ring R and every Baer-ideals I, J of R , $(I \rightarrow J) + (J \rightarrow I) = R$.*

Proof. Let R be a Baer ring and I, J be Baer-ideals of R , then $I \cap J$ is a Baer-ideal. Hence, $R/(I \cap J)$ is a Baer ring and by Proposition 3.1, satisfies BLR-3. It follows as in the proof of Proposition 2.7, that $(I \rightarrow J) + (J \rightarrow I) = R$. \square

Proposition 3.3. 1. *Every quotient (by an ideal) of a multiplication ring is a multiplication ring.*

2. *Every quotient (by an Baer-ideal) of a Baer ring is a multiplication Baer ring.*

Proof. 1. Let R be a multiplication ring and I an ideal of R . Let $J/I \subseteq K/I$ be ideals of R/I , then $J \subseteq K$. Thus, as R is a multiplication ring, there exists an ideal T of R such that $J = K \cdot T$. Hence, $J/I = (K \cdot T)/I = K/I \cdot T/I$. Therefore, R/I is a multiplication ring.

2. [14, Lemma 3] \square

Recall that a Commutative ring with is a Von Neumann ring (VNR) if and only if R_P is a field for all prime ideals of R .

Proposition 3.4. *Every VNR is a multiplication ring.*

Proof. Note that by Proposition 2.2, we simply have to prove BLR-1, which we shall do locally. Let R be a VNR, and I, J be ideals of R , and P a prime ideal of R . We need to show that $I_P \cap J_P = I_P \cdot (I \rightarrow J)_P$. Since R_P is a field, then $I_P = 0$ or R_P and $J_P = 0$ or R_P . The equation is obvious when $I_P = 0$. We consider the remaining two cases.

Case 1: If $I_P = J_P = R_P$, then $J \cap (R \setminus P) \neq \emptyset$. But, $J \subseteq I \rightarrow J$, so $J \cap (R \setminus P) \subseteq (I \rightarrow J) \cap (R \setminus P)$. Hence $(I \rightarrow J) \cap (R \setminus P) \neq \emptyset$ and $(I \rightarrow J)_P = R_P$. Thus $I_P \cdot (I \rightarrow J)_P = R_P \cdot (I \rightarrow J)_P = (I \rightarrow J)_P = R_P = I_P \cap J_P$.

Case 2: Suppose $I_P = R_P$ and $J_P = 0$, then $I \cap (R \setminus P) \neq \emptyset$. We need to show that $(I \rightarrow J)_P = 0$. Since $I \cap (R \setminus P) \neq \emptyset$, there exists $s \in I$ and $s \notin P$. Now, let $x \in (I \rightarrow J)$ and $t \notin P$. Then, $xs \in J$ and since $J_P = 0$, it follows that $xs/t = 0/t$. Thus, $xst' = 0$ for some $t' \notin P$, which implies that $x/t = 0/t$. Whence, $(I \rightarrow J)_P = 0$ as needed. \square

It follows from Proposition 3.4 and its proof that a VNR is a BL-ring if and only if it satisfies for all ideals I, J and all prime ideal P ,

$$I_P = J_P = 0 \text{ implies } (I \rightarrow J)_P = R_P$$

4. REPRESENTATION AND FURTHER PROPERTIES OF BL-RINGS

Recall that it follows from the most celebrated Birkhoff subdirectly irreducible representation Theorem (see for e.g., [1, Thm.8.6]), every BL-ring R

is a subdirect product of subdirectly irreducible rings, all of whom are homomorphic images of R . It follows from Proposition 2.12 that every BL-ring R is a subdirect product of subdirectly irreducible BL-rings. This justifies the need to start our analysis with subdirectly irreducible BL-rings. It is known that for every subdirectly irreducible commutative ring R with minimal ideal M is either R is a field or $M^2 = 0$ (see for e.g., [8]).

Proposition 4.1. *Let R be a subdirectly irreducible BL-ring with minimal ideal M . Then*

1. *The annihilator ideals of R are linearly ordered and finite in number;*
2. *Every ideal of R is either an annihilator ideal or dense;*
3. *M is an annihilator ideal;*
4. *For every annihilator ideal $I \neq R$ and every dense ideal J , $I \subseteq J$ and $J \rightarrow I = I$;*
5. *For every ideals I, J of R , either $I \rightarrow J$ or $J \rightarrow I$ is dense.*

Proof. Note that $A(R)$ is a BL-algebra and $MV(R)$ is an MV-algebra, more precisely the MV-center of $A(R)$. Moreover, $(\bigvee I)^* = \bigcap I^*$, which implies that every subset of $MV(R)$ has an infimum. It follows from this that every subset of $MV(R)$ also has a supremum since $\bigwedge S = (\bigvee S^*)^*$ [6, Lemma 6.6.3]. Thus, $MV(R)$ is a complete MV-algebra. In addition, for every nonzero ideal I of R , $M \subseteq I$, so $I^* \subseteq M^*$. Therefore, for every proper ideal J in $MV(R)$, we have $J \subseteq M^*$. This means that $MV(R) \setminus \{R\}$ has a maximum element, namely M^* (note that $M^* \neq R$ since $M \neq 0$). To see that $(MV(R), \subseteq)$ is a chain, let $X, Y \in MV(R)$ such that $X \not\subseteq Y$ and $Y \not\subseteq X$, then $X \rightarrow Y, Y \rightarrow X \neq R$ and $X \rightarrow Y, Y \rightarrow X \subseteq M^*$. Hence, by the pre-linearity axiom, $R = (X \rightarrow Y) \vee (Y \rightarrow X) \subseteq M^*$. Hence, $M^* = R$, which is a contradiction. Therefore, $(MV(R), \subseteq)$ is an MV-chain as claimed. But the only complete MV-chains are finite Łukasiewicz chains and $[0, 1]$. The condition $MV(R) \setminus \{R\}$ has a maximum element implies that $MV(R)$ is a finite Łukasiewicz chain. This completes the proof of (1) and (3).

On the other hand, since the MV-center of $A(R)$ is a finite Łukasiewicz chain, it follows from [4, Remark 3.3.2] that $A(R) \cong MV(R) \oplus D(R)$, the ordinal sum of $MV(R)$ and $D(R)$. Readers unfamiliar with the ordinal sums of hoops may [4, §3.1] for basic definitions and properties. The property stated in (2) clearly holds in $MV(R) \oplus D(R)$, and therefore in $A(R)$. Similarly, (4) follows from the definitions of the implication \rightarrow in the ordinal sum.

It remains to prove (5). Since $A(R)$ is a BL-algebra, it is known (see for e.g., [4, p. 368]) that the set $D(L)$ of dense elements of any BL-algebra L is an implicative filter and $L/D(R) \cong MV(L)$. Therefore, $A(R)/D(R) \cong MV(R)$,

and since $MV(R)$ is linearly ordered, we deduce that $A(R)/D(R)$ is linearly ordered. The conclusion now follows from the definitions of order and \rightarrow on $A(R)/D(R)$. \square

The following result, which is the analog of [3, Theorem 3.9] shields light on the structure of a general BL-ring.

Theorem 4.2. *(A Representation Theorem for BL-rings) Every BL-ring R is a subdirect product of a family $\{R_t : t \in T\}$ of subdirectly irreducible BL-rings satisfying:*

1. $A(R_t) \cong MV(R_t) \oplus D(R_t)$ for all $t \in T$;
2. $A(R)$ is a subdirect product of $\{A(R_t) : t \in T\}$;
3. $A(R_t)$ is a BL-algebra with a unique atom.

Proof. In light of the opening remarks of this section, more importantly Proposition 4.1 and its proof, we only need to prove (2). To show that $A(R)$ is a subdirect product of $\{A(R_t) : t \in T\}$, recall that there is a family $\{I_t : t \in T\}$ ideals of R such that $R_t = R/I_t$ for all t and $\bigcap I_t = 0$. Now, define $\Theta : A(R) \rightarrow \prod_{t \in T} A(R/I_t)$ by $\Theta(I) = I + I_t \pmod{I_t}$. It is readily verified that Θ is a subdirect embedding of BL-algebras. \square

We shall end our study by establishing some (further) properties of BL-rings.

Proposition 4.3. *Let R be a BL-ring. Then*

- (i) *Let I be an ideal of R and P a prime ideal of R . Then $I \subseteq P$ or $I \rightarrow P = P$.*
- (ii) *Every proper ideal is contained in a prime ideal of R .*
- (iii) *If $P, Q \subseteq R$ are prime ideals that are not comparable, then they are comaximal, that is, $P + Q = R$.*
- (iv) *If $P, Q \subseteq R$ are distinct minimal prime ideals, then they are comaximal.*
- (v) *Let I be an ideal of R . Then the prime ideals below I , if any, form a chain.*
- (vi) *Suppose R is a local ring. Then the prime ideals of R form a chain and each ideal is a power of a prime ideal.*
- (vii) *Suppose $P, Q \subseteq R$ are prime ideals that are not comparable. Then there is a BL-epimorphism from $A(R) \rightarrow A(R/P \oplus R/Q)$.*
- (viii) *Suppose the set of minimal primes $\text{Min}(R)$ is finite, then R is a finite direct sum of Dedekind domains and special primary rings.*

- Proof.* (i) Let I be an ideal of R and P a prime ideal of R . We have from BL-1 $I \cap P = I(I \rightarrow P)$. So $I(I \rightarrow P) \subseteq P$. Thus $I \subseteq P$ or $I \rightarrow P \subseteq P$. Clearly $P \subseteq I \rightarrow P$ and the result follows.
- (ii) This holds since every BL-ring is generated by idempotents (Corollary 2.3 1.). In fact, Let I be a proper ideal R and let $a \notin I$. There is an $e \in R$ with $e^2 = e$ such that $a = ae$. It is clear that $e \notin I$ since $a \notin I$. Let P be a maximal ideal with respect containing I but not containing e . Suppose now that $xRy \subseteq P$ where $x, y \notin P$. then $e \in P + RxR$ and $e \in P + RyR$ (as if e is not in the ideal it would contradict the maximality of P). It follows that $e \in P + xRyR \subseteq P$, which is a contradiction. Thus P is prime.
- (iii) This follows from the combination of BL-2 and (i), since $R = (P \rightarrow Q) + (Q \rightarrow P) = P + Q = R$.
- (iv) This follows from the fact that distinct minimal primes are not comparable and the use of (iii).
- (v) Suppose $P, Q \subseteq I$ with P and Q prime ideals. If P and Q are not comparable, we have $R = P + Q \subseteq R$ which contradicts the fact that I is proper. Hence the prime ideals below I , if any, form a chain.
- (vi) Suppose R is a local ring. We have a unique maximal ideal that contains all the prime ideals of R . Thus the prime ideals of R form a chain by (v). Now the radical of an ideal I of R is the intersection of all prime ideals containing I . Since the prime ideals form a chain, the intersection of a chain of prime ideals is a prime ideal and it then follows that the radical of I is a prime ideal. Since R is a multiplication ring, it follows that I is a power of a prime ideal [13, Theorem 5.]
- (vii) Suppose $P, Q \subseteq R$ are prime ideals that are not comparable. We know that R/P and R/Q are BL-algebras (so $R/p \oplus R/Q$ is also a BL-algebra) and $P + Q = R$. Also, $R^2 + P = R^2 + Q = R$. By the Chinese Remainder Theorem the natural map $R \rightarrow R/p \oplus R/Q$ is onto. Thus the natural map induces naturally a BL-algebra epimorphism $A(R) \rightarrow A(R/p \oplus R/Q)$.
- (viii) This holds since R is a multiplication ring and [13, Theorem 11.].

□

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